

# Using $H_\infty$ for Determination OF the First and Second Order SISO Continuous Systems Types

Muawia Mohamed Ahmed Mahmoud

Head of Control Engineering Department, Faculty of Engineering, Al-Neelain University, Khartoum, Sudan  
Muawia15858@hotmail.com

**Abstract**—There are many known methods used to identify the SISO continuous system if it is over damped or under damped such as finding the damping ratio of the system or plotting its step response to watch the system's behaviour. This paper describes an alternative method using the  $H_\infty$  norm to test the system's behaviour, hence its type.

**Index Terms** —Norm-infinity, Overdamped, Underdamped, Sigma plot.

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## 1. INTRODUCTION

The norm of a dynamic system is the maximum amplification the system can make to the energy of the input signal.  $H_\infty$  is the space of matrix-valued functions that are analytic and bounded in the open right-half of the complex plane (S-Plane). The  $H_\infty$  norm is the maximum singular value of the function over that space. For SISO systems this is interpreted as the maximum magnitude of the frequency response.

Maximum magnitude represents the peak gain of the frequency response.

For a MIMO system, the  $H_\infty$  norm is the peak gain across all input/output channels<sup>[1]</sup>. The aim of the paper is to distinguish the type of the system according to its norm-infinity ( $H_\infty$ ).

## 2 SYSTEM'S ORDER

According to the order of the system, the infinity-norm implies the system behaviour including the maximum singular point and at what frequency it occurs.

### 2.1 First Order.

The Transfer Function of the first order indicates whether the system is Increasing or Decreasing. Increasing system could be classified as High Pass since its amplitude increases with time. The Decreasing first order system could be classified as Low Pass since its amplitude decreases with time<sup>[2]</sup>.

To identify the first order system as a High Pass or Low Pass, the following steps should be done:

Rewrite the system's Transfer Function in the standard form.

Check the Laplace operator's coefficients in the numerator (a) and the denominator (b).

If  $(a) \geq (b)$  the system is High Pass, and if  $(a) < (b)$ , the system is Low Pass.

Assume the Transfer Function  $G(s)$  of a first order system with unity gain is:

$$G(s) = (0.5S + 1) / (S + 1) \quad \dots (1)$$

The Laplace's coefficient of the num is 0.5 whereas the coefficient of the den is 1, thus the system is decreasing (Low Pass).

### 2.1.1 $H_\infty$ of the First Order

For Increasing system i.e  $(a) \geq (b)$  with unity gain, the  $H_\infty$  is equal to  $a/b$ , for Decreasing systems  $H_\infty$  is equal to 1, which means substituting Laplace's operator (s) by zero in the Transfer Function<sup>[3]</sup>.

For the system given in (1), since it is Decreasing, the  $H_\infty$  of the system could be found by substituting Laplace's operator (s) by zero which leads to  $H_\infty = 1$  or 0 dB.

The 0 dB is the maximum singular point of the system at a frequency equal to zero.

Matlab Sigma plot explains this point as shown in figure 1.

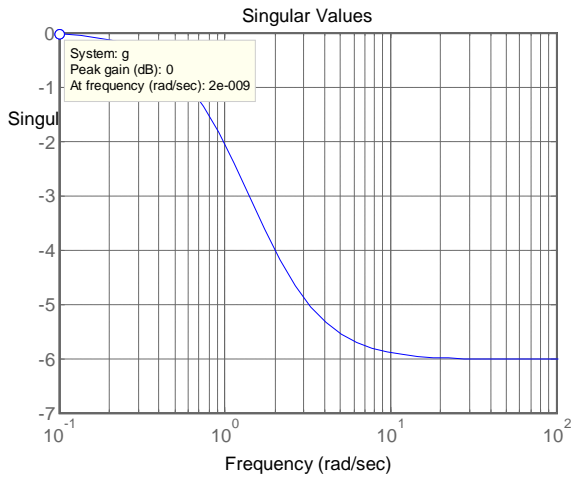


Figure (1): Singular point of decreasing first order system

Note that for decreasing or Low Pass system the  $H_{\infty}$  equals the maximum gain of the system (unity or 0 dB) and appears at zero frequency<sup>[4]</sup>.

For increasing or High Pass system the coefficient of the Laplace operator in the num is bigger than that of the den of the transfer function as shown in (2)

$$G(s) = (2s+1)/(s+1) \dots (2)$$

$H_{\infty}$  of this system equals  $a/b = 2/1 = 2$  or 6.02 dB. The Matlab Sigma plot is shown in figure 2

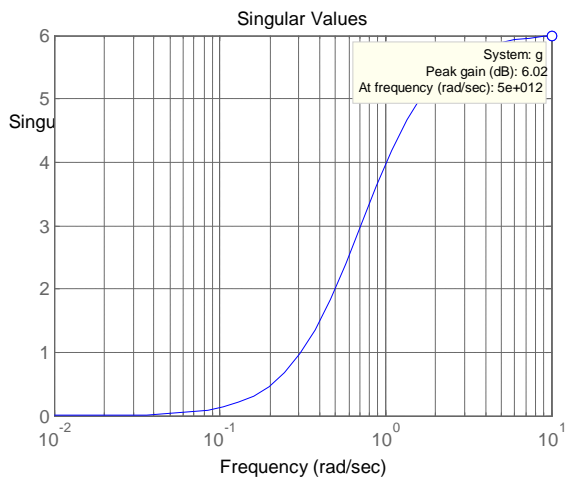


Figure (2): Singular point of increasing first order system

Thus for increasing first order system with unity steady gain, the maximum singular point equals  $a/b$  at the highest possible frequency.

### 2.2 $H_{\infty}$ of Second order Systems

The aim is to find the  $H_{\infty}$  for five cases of the second order system, these are: over damped, under damped, undamped, growing oscillation, and run-away systems.

Growing oscillation and run-away systems indicate that the damping ratio is negative<sup>[5]</sup>.

#### 2.2.1 $H_{\infty}$ of Overdamped 2<sup>nd</sup> order system

An example of such system could be:

$$G(s) = 1/(s^2 + 2s + 1) \dots (3)$$

The  $H_{\infty}$  of this system equals 1. This could be found by using Matlab command:

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>> norm(G(s),inf)
```

At the same time could be found directly by substituting  $S$  in the transfer function by zero. Thus the maximum singular point of the overdamped second order system equals 0 dB at zero frequency.

Sigma plot explains this as shown in figure 3

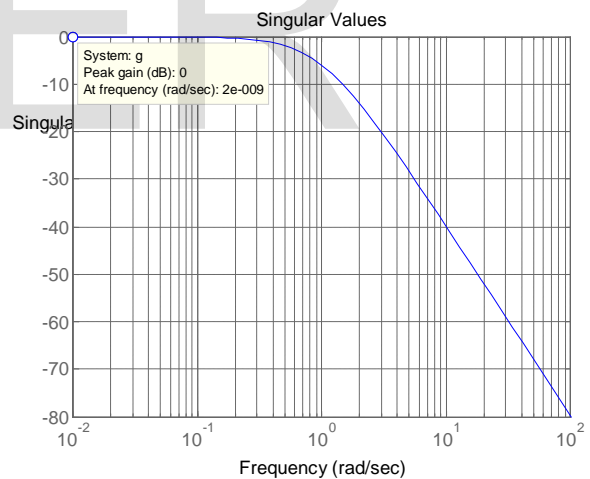


Figure (3): Singular point of 2<sup>nd</sup> order over damped system

#### 2.2.2 $H_{\infty}$ of Underdamped 2<sup>nd</sup> order system

The underdamped system could be described by the following transfer function:

$$G(s) = 1/(s^2 + 0.5s + 1) \dots (4)$$

Since there exists a natural frequency in underdamped systems, computing  $H_{\infty}$  by substituting  $S$  by zero is

invalid. Using Matlab the  $H_\infty$  of this system equals 2.1 or 6.44 dB at a frequency of 0.935 rad/s as shown in figure 4.

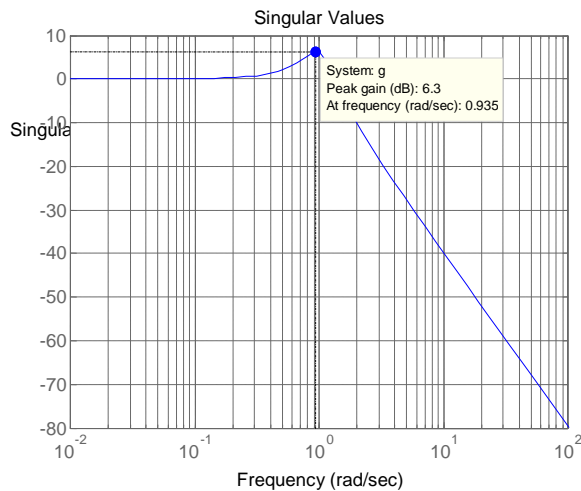


Figure 4 Singular point of 2<sup>nd</sup> order underdamped system

### 2.2.3 $H_\infty$ of Undamped 2<sup>nd</sup> order system

An example of the transfer function of this type of second order could be:

$$G(s) = 1/(s^2 + 1) \quad \dots (5)$$

Note the zero damping ratio.

$H_\infty$  of this system always equals infinity at unity rad/s frequency. The sigma plot is shown in figure 5.

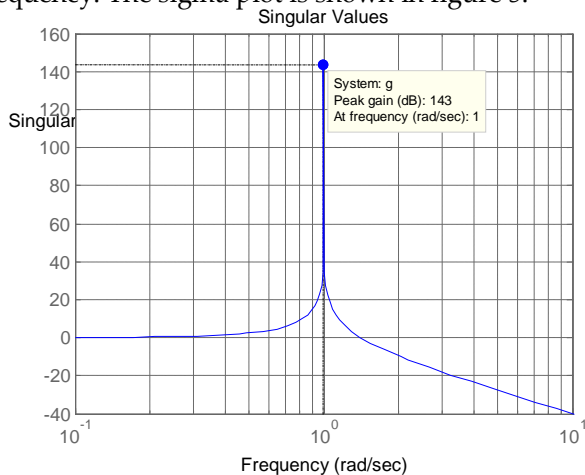


Figure (5): Singular point of Undamped 2<sup>nd</sup> order system

Note that the frequency of the maximum gain occurs at the square root of the system pole, or;

$$\omega = \sqrt{P} \quad \dots (6)$$

p is absolute value of the systems's pole.

### 2.2.4 $H_\infty$ of Unstable 2<sup>nd</sup> order system

According to the step response, the unstable second order systems are classified into two types, growing oscillation system, and run-away system. As known the damping ratio ( $\zeta$ ) for both is less than one. If  $-1 < \zeta < 0$ , the system is unstable with growing oscillation. If  $\zeta \leq -1$ , the system is a run-away second order system<sup>[6]</sup>.

An example of the growing oscillation second order system's transfer function could be:

$$G(s) = 1/(s^2 - s + 1) \quad \dots (7)$$

For this example the damping ratio ( $\zeta$ ) equals -0.5 and the system is unstable with growing oscillation. Matlab computes the value of  $H_\infty$  for this system as 1.15 (1.24 dB).

The sigma plot is shown in figure (6).

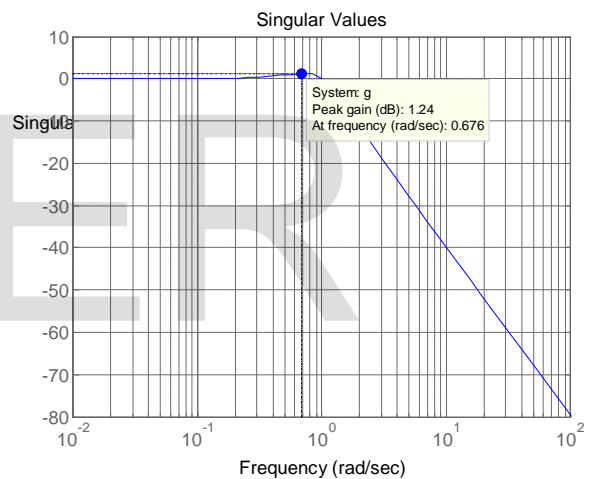


Figure (6): Singular point of unstable growing oscillation system.

Since there exists an oscillation, the maximum overshoot or singular point frequency is a real value (0.676 rad/s).

The transfer function

$$G(s) = 1/(s^2 - 4s + 1) \quad \dots (8)$$

represents an unstable run-away second order system. The  $H_\infty$  for this type of systems could be found directly by substituting S in the transfer function by zero.

The sigma plot of this type of systems is shown in figure (7).

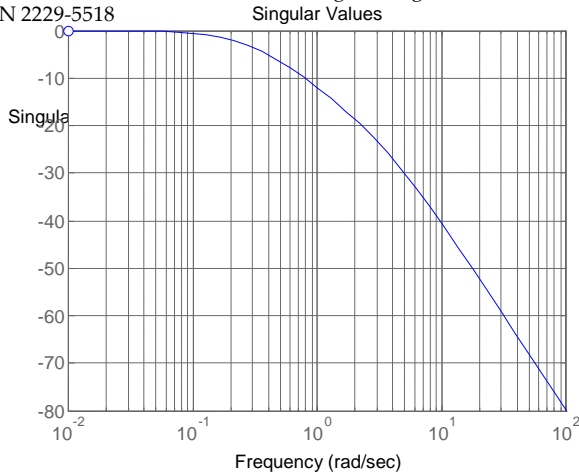


Figure (7): Singular point of the run-away system

### 3. Results

Section 2 discussed the different cases of  $H_\infty$  for first and second order continuous systems expressed in a standard form.

These results are summarised in table 1 below.

a/b in the table means dividing the num S coefficient (a) by the den S coefficient (b).

When  $H_\infty = 1$  in the table, directly substituting S in the transfer function by zero yields unity infinity norm.

The word compute in the table means use Matlab to compute its  $H_\infty$ . The infinity norm of this type of systems has different values of norm infinity.

Table 1.  $H_\infty$  of different types of systems

System's order	$H_\infty$	Type	Notes
First	a/b	Increasing	High Pass
First	1	Decreasing	Low Pass
Second	1	Over damped	$\zeta \geq 1$
Second	Compute	Underdamped	$0 < \zeta < 1$
Second	Inf	Undamped	$\omega = \sqrt{P}$
Second	Compute	Growing Oscillation	$-1 < \zeta < 0$
Second	1	Run-away	$\omega = 0$

### 4. CONCLUSION

In this paper the infinity norm was used to determine the type of the first and second order system. As the step response is used to check the behaviour of the system, the  $H_\infty$  method will be an alternative to test the type of the system.

One of the main benefits of associating the infinity norm with the system's type is that from the step response of the system, table 1 gives directly the infinity norm of that system.

The case of  $H_\infty = 1$  in the first order system indicates that it is decreasing or Low Pass system.

The two cases of the second order systems in which the infinity norm equals 1, are either the system is overdamped, or run-away system.

In all cases of the unity infinity norm, substituting  $s=0$  in the transfer function yields this unity infinity norm.

### References

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